National Technical University of Athens
School of Electrical and Computer Engineering

Bistability in Nanorod

Static Magnetic and Electric fields

Problem 1

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Introduction

When an infinite line charge is surrounded by a material whose permittivity depends in a specific way on the electric field, a very interesting phenomenon occurs: bistability. This phenomenon is encountered in nanophotonic devices and forms the basis for optical switches and memory.

In this work, we will study this phenomenon for an infinite line charge λ enclosed by a cylindrical shell of radius α made of a material with permittivity: $\varepsilon(\mathbf{E}) = 1 + \frac{\gamma}{1+\kappa|\mathbf{E}|^2}$

We will examine the role of permittivity ε, the saturation parameter γ, the excitation field E_0, and the nonlinearity parameter κ in the characteristics of bistability and under what conditions they lead to it, both through graphs and interactive tools.

Contents

1 Theory

To study this problem, we need Gauss's law:

$$\oint_{\mathcal{S}} \varepsilon(\mathbf{E})\, \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\varepsilon_0} \tag{1}$$

We also need the fact that for line charges:

$$Q = \lambda h \tag{2}$$

2 Mathematical Solution

2.1 Electric Field Analysis

2.1.1 Region r<α

$$\oint_{\mathcal{S}} \varepsilon(\mathbf{E})\, \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}$$

Due to cylindrical symmetry, the field has only a radial component: $\mathbf{E} = E(r)\,\hat{r}$. We choose a cylindrical Gaussian surface of radius r and height h and substitute (2):

$$\oint_{\mathcal{S}} \varepsilon(E(r))\, E(r)\hat{r} r\, d\phi\, dz\hat{r} = \frac{\lambda h}{\varepsilon_0}$$

Substituting the permittivity:

$$\oint_{\mathcal{S}} E(r) r\hat{r}\, d\phi\, dz\hat{r} \; + \oint_{\mathcal{S}} \frac{\gamma E(r) r\hat{r}}{1+\kappa|E(r)|^2}\, d\phi\, dz\hat{r} = \frac{\lambda h}{\varepsilon_0}$$

Expanding the surface integral and moving constants outside:

$$E(r)\, r\, \hat{r} \int_{z_0}^{z_0+h}\int_0^{2\pi} d\phi\, dz\, \hat{r} + \frac{\gamma E(r) r\hat{r}}{1+\kappa|E(r)|^2} \int_{z_0}^{z_0+h}\int_0^{2\pi} d\phi\, dz\, \hat{r} = \frac{\lambda h}{\varepsilon_0}$$

Solving the integrals and simplifying:

$$2\pi h E(r) r \left(1 + \frac{\gamma}{1+\kappa|E(r)|^2}\right) = \frac{\lambda h}{\varepsilon_0}$$

Expanding yields the final cubic equation:

$$\kappa\varepsilon_0 2\pi r E^3(r) - \lambda\kappa E^2(r) + (\varepsilon_0 2\pi r + \varepsilon_0 2\pi r\gamma)E(r) - \lambda = 0 \tag{3}$$

2.1.2 Region r>α

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\varepsilon_0}$$

Due to cylindrical symmetry, the field has only a radial component: $\mathbf{E} = E(r)\,\hat{r}$. We choose a cylindrical Gaussian surface of radius r and height h and substitute (2):

$$\oint_S E(r)\hat{r} r\, d\phi\, dz \hat{r} = \frac{\lambda h}{\varepsilon_0}$$

Expanding the surface integral and moving constants outside:

$$E(r)\, r\, \hat{r} \int_{z_0}^{z_0+h} \int_0^{2\pi} d\phi\, dz\, \hat{r} = \frac{\lambda h}{\varepsilon_0}$$

Solving the integrals and simplifying:

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{4}$$

2.2 Conditions for Bistability

For bistability, the field needs to have more than one solution. Equation (4) is immediately rejected.

First condition: r<α For the case where r<α we have equation (3), which needs to have more than one solution. This occurs when the discriminant is greater than or equal to zero.

2.2.1 Discriminant Analysis of Cubic Electric Field Equation

The discriminant has the form:

$$\Delta = 18\varepsilon_0^4\kappa^2 E_0^2(1+\gamma)\frac{\alpha^2}{r^2} - 4\varepsilon_0^4 E_0^4\kappa^3\frac{\alpha^4}{r^4} + E_0^2\kappa^2\varepsilon_0^4(1+\gamma)^2\frac{\alpha^2}{r^2} - 4\varepsilon_0^4\kappa(1+\gamma)^3 - 27\varepsilon_0^4\kappa^2 E_0^2\frac{\alpha^2}{r^2}$$

Where $E_0 = \frac{\lambda}{2\pi\varepsilon_0\alpha}$. With $E_1 = \kappa E_0^2\frac{\alpha^2}{r^2}$ we need:

$$-4E_1^2 + ((1+\gamma)^2 + 18(1+\gamma) - 27)E_1 - 4(1+\gamma)^3 \geq 0 \tag{5}$$

For this to hold, at least one solution must exist since the quadratic is concave. Therefore we need $\Delta_2 \geq 0$. Taking the limiting case $\Delta_2 = 0$:

$$\begin{gathered}((1+\gamma)^2 + 18(1+\gamma) - 27)^2 - 64(1+\gamma)^3 = 0 \\ \Rightarrow (\gamma - 8)^3(\gamma + 4) = 0 \\ \Rightarrow \gamma = 8\,(\gamma \geq 0)\end{gathered}$$

Second condition for bistability: $\gamma \geq 8$

2.3 Conclusion

For the bistability phenomenon to occur, we need:

1. $r \leq \alpha$
2. $\gamma \geq 8$

3 Study of the Phenomenon

Having established the conditions and mathematical tools, we can now process the data and better understand this phenomenon.

3.1 Existence of Bistability

- By solving (5), we find the interval where bistability exists.
- The two solutions of the equation at 0 give the upper bound and lower bound for r=a.
- We observe that there is a critical point $(8, 27)$ where bistability begins to exist.

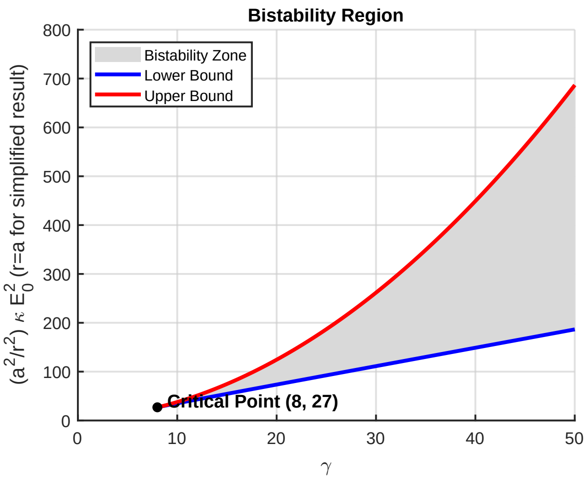

Figure 1: Bistability region

3.2 Influence of E_0 on Bistability Region Size

To study the influence of E_0^2, we keep all other parameters constant except E_0^2 and the normalized distance $\frac{r}{\alpha}$.

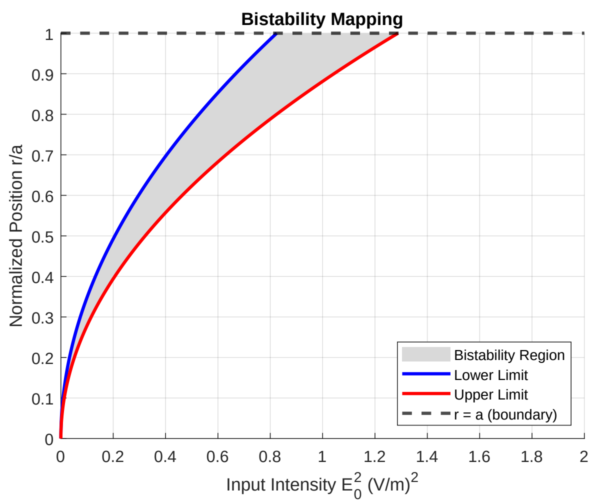

Figure 2: Bistability region for $\gamma = 18$ and $\kappa = 80\ (m/V)^2$

- The region grows parabolically.
- There is no bistability for $\frac{r}{\alpha} > 1$, i.e., for $r > \alpha$.

The relationship $E_0 = f(r)$ is of great interest when the other parameters are constant.

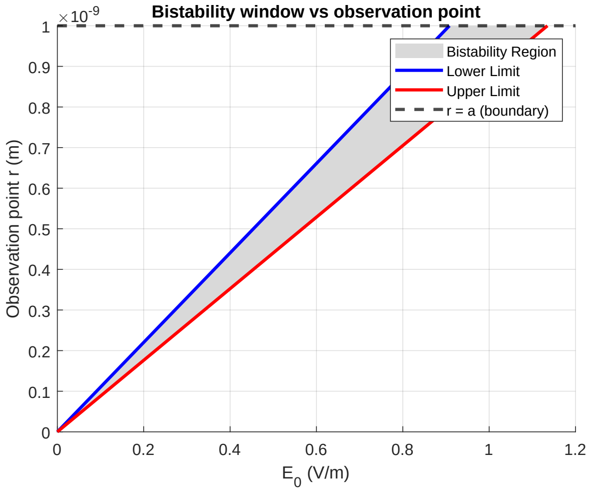

Figure 3: Bistability region for $\alpha = 1\ nm$, $\gamma = 18$ and $\kappa = 80\ (m/V)^2$

- Despite the complexity of the system, the relationship is linear.

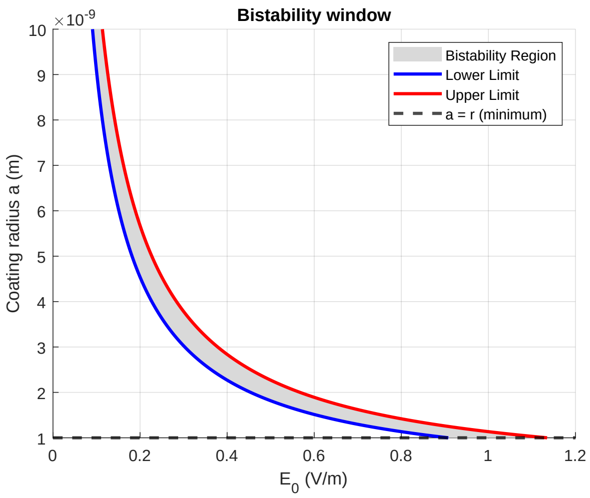

Figure 4: Bistability region for $r = 1\ nm$, $\gamma = 18$ and $\kappa = 80\ (m/V)^2$

- The relationship is inversely linear.
- There is a point where the bistability region is maximum.

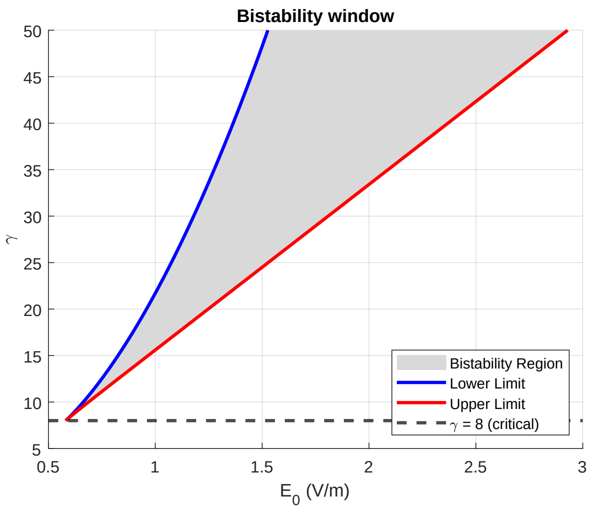

Figure 5: Bistability region for $\frac{r}{\alpha} = 1$ and $\kappa = 80\ (m/V)^2$

- The larger γ is, the larger the bistability region.

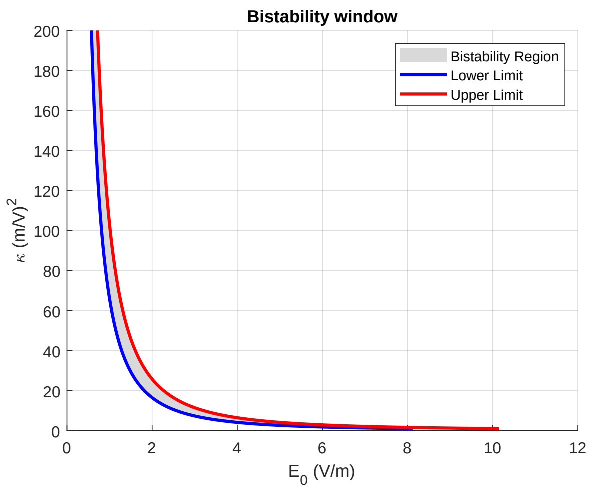

Figure 6: Bistability region for $\frac{r}{\alpha} = 1$ and $\gamma = 18$

- Similarly to parameter α, there is a point where the bistability region is maximum.

3.3 Memory Property

The value of $E(r)$ depends on the past, that is:

- If E_0 was **increasing** before reaching the region, it moves between the blue square and red triangle. ($0.063\ V/m \leq E(r) \leq 0.112\ V/m$)
- If E_0 was **decreasing** before reaching the region, it moves between the cyan square and pink triangle. ($0.469\ V/m \leq E(r) \leq 0.861\ V/m$)

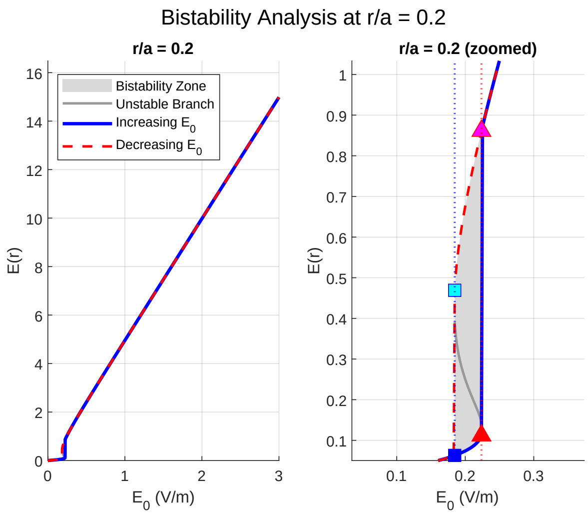

Figure 7: $E(r)$ for $\frac{r}{\alpha} = 0.2$, $\gamma = 18$, $\kappa = 80\ (m/V)^2$

- If E_0 was **increasing** before reaching the region, it moves between the blue square and red triangle. ($0.063\ V/m \leq E(r) \leq 0.118\ V/m$)
- If E_0 was **decreasing** before reaching the region, it moves between the cyan square and pink triangle. ($0.456\ V/m \leq E(r) \leq 0.867\ V/m$)

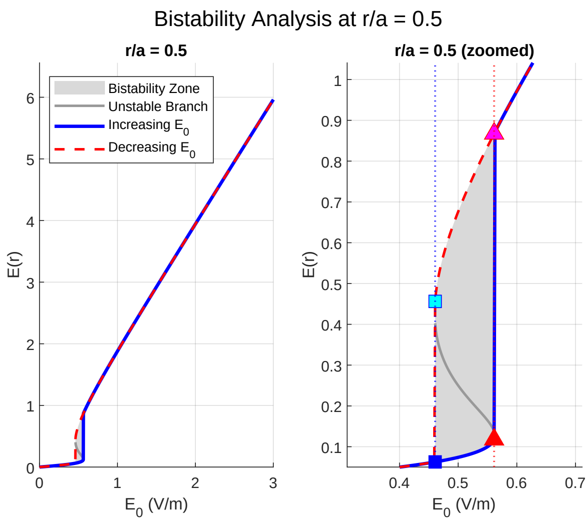

Figure 8: $E(r)$ for $\frac{r}{\alpha} = 0.5$, $\gamma = 18$, $\kappa = 80\ (m/V)^2$

- If E_0 was **increasing** before reaching the region, it moves between the blue square and red triangle. $(0.063\ V/m \leq E(r) \leq 0.119\ V/m)$
- If E_0 was **decreasing** before reaching the region, it moves between the cyan square and pink triangle. $(0.460\ V/m \leq E(r) \leq 0.869\ V/m)$

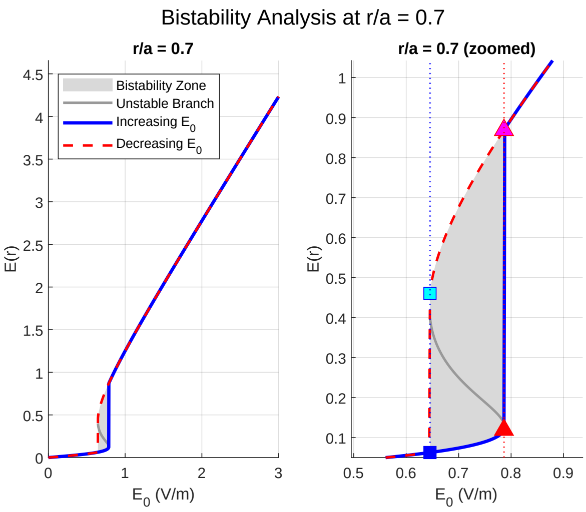

Figure 9: $E(r)$ for $\frac{r}{\alpha} = 0.7$, $\gamma = 18$, $\kappa = 80\ (m/V)^2$

- If E_0 was **increasing** before reaching the region, it moves between the blue square and red triangle. $(0.063\ V/m \leq E(r) \leq 0.120\ V/m)$
- If E_0 was **decreasing** before reaching the region, it moves between the cyan square and pink triangle. $(0.452\ V/m \leq E(r) \leq 0.869\ V/m)$

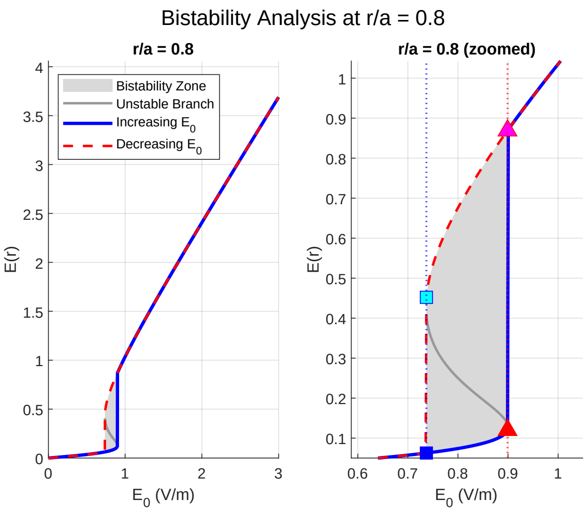

Figure 10: $E(r)$ for $\frac{r}{\alpha} = 0.8$, $\gamma = 18$, $\kappa = 80\ (m/V)^2$

3.3.1 Observations

We observe that the field ”remembers” where it was, resulting in the direction of E_0 variation playing a role.

We also observe that the larger r/α is, the larger the value of E_0 needed to enter the bistability region, but at the same time, the wider this region becomes.

3.4 Electric Field vs. Normalized Distance Characteristics

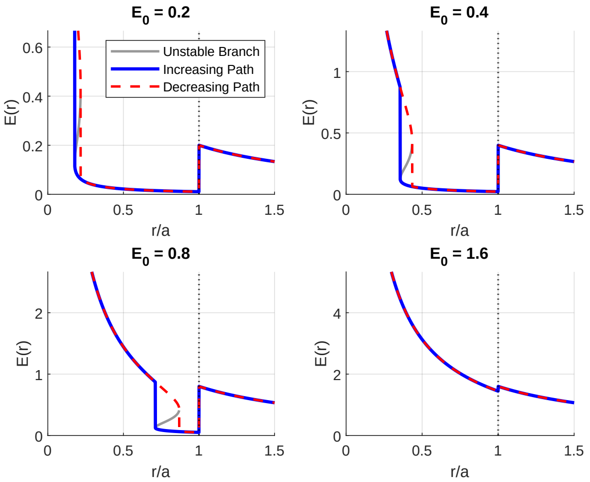

Figure 11: Characteristic $E = f(r/\alpha)$, $\gamma = 18$ and $\kappa = 80$ $(m/V)^2$

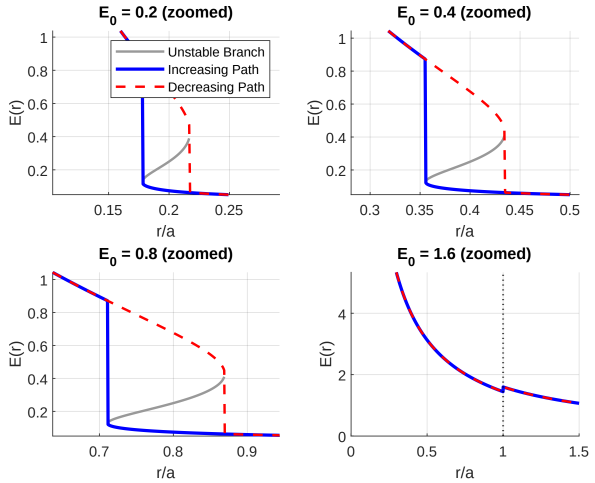

Figure 12: Characteristic $E = f(r/\alpha)$, focused on bistability region

Due to the finite radius of the cylinder, at point $r/\alpha = 1$ there is a discontinuity, because the field inside the cylindrical shell is not the same as the field outside it.

We observe similar behavior to the previous subsection. As the variable E_0 increases, the bistability region grows and shifts to the right, maintaining its memory property.

A difference appears in the graph for $E_0 = 1.6$ V/m, because the bistability region has shifted so far to the right that it lies outside the cylindrical shell and therefore does not appear.

4 Applications

Using the above mathematical relations, we can create the following tools:

4.1 Bistability in Nanorod Calculation Tool

```
--- Thickness Calculator ---
Enter gamma: 18
Enter kappa: 50
Enter target depth for ignition (r in nm): 1
Enter excitation field (E0 in V/m): 0.5

Required coating radius: a        = 2.8463 nm
Snap-up field:          E_snap_up   = 1.4232 V/m
Snap-down field:        E_snap_down = 1.1639 V/m
Hysteresis window:      dE0         = 0.2593 V/m
```

Figure 13: Input and output

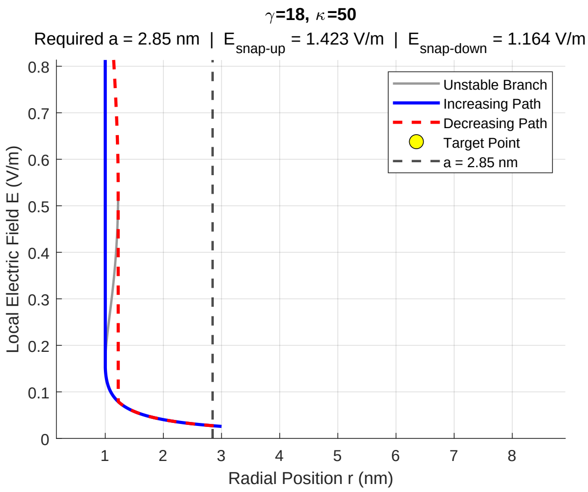

Figure 14: Output graph

With simple input of parameters, we can calculate the thickness of the cylindrical shell so that the bistability phenomenon appears at the desired distance.

4.2 Bistability Visualization Tool

Using sliders, we can change the system parameters in real time and observe their effect on the bistability graph. This allows us to easily develop intuition for the phenomenon.